A CENTRAL LIMIT THEOREM FOR FIELDS OF MARTINGALE DIFFERENCES

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Abstract. We prove a central limit theorem for stationary random fields of martingale differences $f \circ T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, where $T_{\underline{i}}$ is a $\mathbb{Z}^d$ action and the martingale is given by a commuting filtration. The result has been known for Bernoulli random fields; here only ergodicity of one of commuting transformations generating the $\mathbb{Z}^d$ action is supposed.

Introduction

In study of the central limit theorem for dependent random variables, the case of martingale difference sequences has played an important role, cf. Hall and Heyde, [HaHe]. Limit theorems for random fields of martingale differences were studied for example by Basu and Dorea [BD], Morkvenas [M], Nahapetian [N], Poghosyan and Roelly [PR], Wang and Woodroofe [WaW]. Limit theorems for martingale differences enable a research of much more complicated processes and random fields. The method of martingale approximations, often called Gordin's method, originated by Gordin's 1969 paper [G1]. The approximation is possible for random fields as well, for most recent results cf. e.g. [WaW] and [VWa]. Remark that another approach was introduced by Dedecker in [D] (and is being used since); it applies both to sequences and to random fields.

For random fields, the martingale structure can be introduced in several different ways. Here we will deal with a stationary random field $f \circ T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, where f is a measurable function on a probability space $(\Omega, \mu, \mathcal{A})$ and $T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, is a group of commuting probability preserving transformations of $(\Omega, \mu, \mathcal{A})$ (a $\mathbb{Z}^d$ action). By $e_i \in \mathbb{Z}^d$ we denote the vector $(0, ..., 1, ..., 0)$ having 1 on the i-th place and 0 at all other places, $1 \le i \le d$.

$\mathcal{F}_{\underline{i}}$, $\underline{i} = (i_1 \dots, i_d) \in \mathbb{Z}^d$, is an invariant commuting filtration (cf. D. Khosnevisan, [K]) if

(i) $\mathcal{F}_{\underline{i}} = T^{-\underline{i}}\mathcal{F}_{\underline{0}}$ for all $\underline{i} \in \mathbb{Z}^d$,

(ii) $\mathcal{F}_{\underline{i}} \subset F_{\underline{j}}$ for $\underline{i} \le \underline{j}$ in the lexicographic order, and

(iii) $\mathcal{F}_{\underline{i}} \cap \mathcal{F}_{\underline{j}} = \mathcal{F}_{\underline{i} \wedge \underline{j}}$, $\underline{i}, \underline{j} \in \mathbb{Z}^d$, and $\underline{i} \wedge \underline{j} = (\min\{i_1, j_1\}, \dots, \min\{i_d, j_d\})$.

If, moreover, $E\Big(E(f|\mathcal{F}_{\underline{i}})\big|\mathcal{F}_{\underline{j}}\Big) = E(f|\mathcal{F}_{\underline{i} \wedge \underline{j}})$, for every integrable function f, we say that the filtration is *completely commuting* (cf. [G2], [VWa]).

By $\mathcal{F}_l^{(q)}$, $1 \le q \le d$, $l \in \mathbb{Z}$, we denote the σ-algebra generated by the union of all

$\mathcal{F}_{\underline{i}}$ with $i_q \le l$. For $d = 2$ we by $\mathcal{F}_{\infty,j} = \mathcal{F}_j^{(2)}$ denote the σ-algebra generated by the union of all $\mathcal{F}_{i,j}$, $i \in \mathbb{Z}$, and in the same way we define $\mathcal{F}_{i,\infty}$.

We sometimes denote $f \circ T_{\underline{i}}$ by $U_{\underline{i}} f$; f will always be from $\mathcal{L}^2$.

We say that $U_{\underline{i}} f$, $\underline{i} \in \mathbb{Z}^d$, is a *field of martingale differences* if f is $\mathcal{F}_{\underline{0}}$-measurable and whenever $\underline{i} = (i_1 \dots, i_d) \in \mathbb{Z}^d$ is such that $i_q \le 0$ for all $1 \le q \le d$ and at least one inequality is strict then $E(f \,|\, \mathcal{F}_{\underline{i}}) = 0$.

Notice that $U_{\underline{i}} f$ is then $\mathcal{F}_{\underline{i}}$-measurable, $\underline{i} = (i_1 \dots, i_d) \in \mathbb{Z}^d$, and if $\underline{j} = (j_1 \dots, j_d) \in \mathbb{Z}^d$ is such that $j_k \le i_k$ for all $1 \le k \le n$ and at least one inequality is strict, $E(U_{\underline{i}} f \,|\, \mathcal{F}_{\underline{j}}) = 0$.
Notice that by commutativity, if $U_{\underline{i}} f$ are martingale differences then

$$E(f \,|\, \mathcal{F}_{-1}^{(q)}) = 0$$

for all $1 \le q \le d$. $(f \circ T_{e_q}^j)_j$ is thus a sequence of martingale differences for the filtration of $\mathcal{F}_j^{(q)}$. In particular, for $d = 2$, $(f \circ T_{e_2}^j)$ is a sequence of martingale differences for the filtration of $\mathcal{F}_{\infty,j} = \mathcal{F}_j^{(2)}$.

Recall that a measure preserving transformation T of $(\Omega, \mu, \mathcal{A})$ is said to be *ergodic* if for any $A \in \mathcal{A}$ such that $T^{-1}A = A$, $\mu(A) = 0$ or $\mu(A) = 1$. Similarly, a $\mathbb{Z}^d$ action $(T_{\underline{i}})_{\underline{i}}$ is ergodic if for any $A \in \mathcal{A}$ such that $T_{-\underline{i}} A = A$, $\mu(A) = 0$ or $\mu(A) = 1$.

A classical result by Billinsley and Ibragimov says that if $(f \circ T^i)_i$ is an ergodic sequence of martingale differences, the central limit theorem holds. The result does not hold for random fields, however.

Example. As noticed in paper by Wang, Woodroofe [WaW], for a 2-dimensional random field $Z_{i,j} = X_i Y_j$ where X_i and Y_j, $i, j \in \mathbb{Z}$, are mutually independent $\mathcal{N}(0,1)$ random variables, we get a convergence towards a non normal law. The random field of $Z_{i,j}$ can be represented by a non ergodic action of $\mathbb{Z}^2$:

Let $(\Omega, \mu, \mathcal{A})$ be a product of probability spaces $(\Omega', \mu', \mathcal{A}')$ and $(\Omega'', \mu'', \mathcal{A}'')$ equipped with ergodic measure preserving transformations T' and T''. On Ω we then define a measure preserving $\mathbb{Z}^2$ action $T_{i,j}(x,y) = (T'^i x, T''^j y)$. The σ-algebras $\mathcal{A}', \mathcal{A}''$ are generated by $\mathcal{N}(0,1)$ iid sequences of random variables $(e' \circ T'^i)_i$ and $(e'' \circ T''^i)_i$ respectively. The dynamical systems $(\Omega', \mu', \mathcal{A}', T')$ and $(\Omega'', \mu'', \mathcal{A}'', T'')$ are then Bernoulli hence ergodic (cf. [CSF]). On the other hand, for any $A' \in \mathcal{A}'$, $A' \times \Omega''$ is $T_{0,1}$-invariant hence $T_{0,1}$ is not an ergodic transformation. Similarly we get that $T_{1,0}$ is not an ergodic transformation either. By ergodicity of T', T'', $A' \times \Omega''$, $A' \in \mathcal{A}'$, are the only $T_{0,1}$-invariant measurable subsets of Ω and $A'' \times \Omega'$, $A'' \in \mathcal{A}''$, are the only $T_{1,0}$-invariant measurable subsets of Ω (modulo measure μ). Therefore, the only measurable subsets of Ω which are invariant both for $T_{0,1}$ and for $T_{1,0}$ are of measure 0 or of measure 1, i.e. the $\mathbb{Z}^2$ action $T_{i,j}$ is ergodic.
On Ω we define random variables X, Y by $X(x,y) = e'(x)$ and $Y(x,y) = e''(y)$. The random field of $(XY) \circ T_{i,j}$ then has the same distribution as the random field of $Z_{i,j} = X_i Y_j$ described above. The natural filtration of $\mathcal{F}_{i,j} = \sigma\{(XY) \circ T_{i',j'} : i' \le i, j' \le j\}$ is commuting and $((XY) \circ T_{i,j})_{i,j}$ is a field of martingale differences.

A very important particular case of a $\mathbb{Z}^d$ action is the case when the σ-algebra $\mathcal{A}$ is generated by iid random variables $U_{\underline{i}} e$, $\underline{i} \in \mathbb{Z}^d$. The σ-algebras $\mathcal{F}_{\underline{j}} = \sigma\{U_{\underline{i}} : i_k \le j_k, k = 1, \dots, d\}$ are then a completely commuting filtration and if $U_{\underline{i}} f$, $\underline{i} \in \mathbb{Z}^d$ is a

martingale difference random field, the central limit theorem takes place (cf. [WW]). This fact enabled to prove a variety of limit theorems by martingale approximations (cf. e.g. [WaW], [VWa]).
For Bernoulli random fields, other methods of proving limit theorems have been used, cf. e.g. [ElM-V-Wu], [Wa], [BiDu].

The aim of this paper is to show that for a martingale difference random field, the CLT can hold under assumptions weaker than Bernoullicity.

Main result

Let $T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, be a $\mathbb{Z}^d$ action of measure preserving transformations on $(\Omega, \mathcal{A}, \mu)$, $(\mathcal{F}_{\underline{i}})_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, be a commuting filtration. By $e_i \in \mathbb{Z}^d$ we denote the vector $(0, ..., 1, ..., 0)$ having 1 on the i-th place and 0 at all other places, $1 \le i \le d$.

Theorem. *Let $f \in L^2$, be such that $(f \circ T_{\underline{i}})_{\underline{i}}$ is a field of martingale differences for a completely commuting filtration $\mathcal{F}_{\underline{i}}$. If at least one of the transformations T_{e_i}, $1 \le i \le d$, is ergodic then the central limit theorem holds, i.e. for $n_1, \dots, n_d \to \infty$ the distributions of*

$$\frac{1}{\sqrt{n_1 \dots n_d}} \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} f \circ T_{(i_1,\dots,i_d)}$$

weakly converge to $\mathcal{N}(0, \sigma^2)$ where $\sigma^2 = \|f\|_2^2$.

<u>*Remark 1.*</u> The results from [VoWa] remain valid for $\mathbb{Z}^d$ actions satisfying the assumptions of the Theorem, Bernoullicity thus can be replaced by ergodicity of one of the transformations T_{e_i}. Under the assumptions of the Theorem we thus also get a weak invariance principle. [VoWa] implies many earlier results, cf. references therin and in [WaW].

Proof.

We prove the theorem for $d = 2$. Proof of the general case is similar.
We suppose that the transformation $T_{0,1}$ is ergodic and $\|f\|_2 = 1$. To prove the central limit theorem for the random field it is sufficient to prove that for $m_k, n_k \to \infty$ as $k \to \infty$,

$$\frac{1}{\sqrt{m_k n_k}} \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} f \circ T_{i,j} \text{ converge in distribution to } \mathcal{N}(0,1). \tag{1}$$

Recall the central limit theorem by D.L. McLeish (cf. [M]) saying that if $X_{n,i}$, $i = 1, \dots, k_n$, is an array of martingale differences such that

(i) $\max_{1 \le i \le k_n} |X_{n,i}| \to 0$ in probability,
(ii) there is an $L < \infty$ such that $\max_{1 \le i \le k_n} X_{n,i}^2 \le L$ for all n, and
(iii) $\sum_{i=1}^{k_n} X_{n,i}^2 \to 1$ in probability,

then $\sum_{i=1}^{k_n} X_{n,i}$ converge to $\mathcal{N}(0,1)$ in law.

Next, we will suppose $k_n = n$; we will denote $U_{i,j} f = f \circ T_{i,j}$. For a given positive integer v and positive integers u, n define

$$F_{i,v} = \frac{1}{\sqrt{v}} \sum_{j=1}^{v} U_{i,j} f, \quad X_{n,i} = \frac{1}{\sqrt{n}} F_{i,v}, \quad i = 1, \dots, n.$$

Clearly, $X_{n,i}$ are martingale differences for the filtration $(\mathcal{F}_{i,\infty})_i$. We will verify the assumptions of McLeish's theorem.

The conditions (i) and (ii) are well known to follow from stationarity. For reader's convenience we recall their proofs.

(i) For $\epsilon > 0$ and any integer $v \geq 1$,

$$\mu(\max_{1\leq i\leq n} |X_{n,i}| > \epsilon) \leq \sum_{i=1}^{n} \mu(|X_{n,i}| > \epsilon) = n\mu\Big(\Big|\frac{1}{\sqrt{nv}}\sum_{j=1}^{v} U_{0,j}f\Big| > \epsilon\Big) \leq$$

$$\leq \frac{1}{\epsilon^2} E\Big(\Big(\frac{1}{\sqrt{v}}\sum_{j=1}^{M} U_{0,j}f\Big)^2 1_{|\sum_{j=1}^{v} U_{0,j}f|\geq\epsilon\sqrt{nv}}\Big) \to 0$$

as $n \to \infty$; this proves (i). Notice that that the convergence is uniform for all v.

To see (ii) we note

$$\Big(\max_{1\leq i\leq n} |X_{n,i}|\Big)^2 \leq \sum_{i=1}^{n} X_{n,i}^2 = \frac{1}{n}\sum_{i=1}^{n}\Big(\frac{1}{\sqrt{v}}\sum_{j=1}^{v} U_{i,j}f\Big)^2$$

which implies $E\Big(\max_{1\leq i\leq n} |X_{n,i}|\Big)^2 \leq 1$.

[WaW]

It remains to prove (iii).

Let us fix a positive integer m and for constants $a_1, \dots, a_m$ consider the sums

$$\sum_{i=1}^{m} a_i \sum_{j=1}^{v} U_{i,j}f, \quad v \to \infty.$$

Then $(\sum_{i=1}^{m} a_i U_{i,j}f)_j$, $j = 1, 2, \dots$, are martingale differences for the filtration $(\mathcal{F}_{\infty,j})_j$ and by the central limit theorem of Billingsley and Ibragimov [Bil], [I] (we can also prove using the McLeish's theorem)

$$\frac{1}{\sqrt{v}}\sum_{j=1}^{v}\big(\sum_{i=1}^{m} a_i U_{i,j}f\big)$$

converge in law to $\mathcal{N}(0, \sum_{i=1}^{m} a_i^2)$. Notice that that here we use the assumption of ergodicity of $T_{0,1}$.

From this it follows that the random vectors $(F_{1,v}, \dots, F_{m,v})$ where

$$F_{u,v} = \frac{1}{\sqrt{v}}\sum_{j=1}^{v} U_{u,j}f, \quad u = 1, \dots, m,$$

converge in law to a vector $(W_1, \dots, W_m)$ of m mutually independent and $\mathcal{N}(0,1)$ distributed random variables. For a given $\epsilon > 0$, if $m = m(\epsilon)$ is sufficiently big then we have $\big\|1 - (1/m)\sum_{u=1}^{m} F_{u,v}^2\big\|_1 < \epsilon/2$. Using a truncation anrgument we can from the convergence in law of $(F_{u,v}, \dots, F_{m,v})$ towards $(W_1, \dots, W_m)$ deduce that for $m = m(\epsilon)$ sufficiently big and v bigger than some $v(m, \epsilon)$,

$$\Big\|1 - \frac{1}{m}\sum_{u=1}^{m} F_{u,v}^2\Big\|_1 < \epsilon.$$

Any positive integer N can be expressed as $N = pm + q$ where $0 \le q \le m-1$. Therefore

$$(2) \qquad 1 - \frac{1}{N}\sum_{i=1}^{N} F_{i,v}^2 = \frac{m}{N}\sum_{k=0}^{p-1}\Big(\frac{1}{m}\sum_{i=km+1}^{(k+1)m} F_{i,v}^2 - 1\Big) + \frac{1}{N}\sum_{i=mp+1}^{N} F_{i,v}^2 - \frac{q}{N}.$$

There exists an N_ϵ such that for $N \ge N_\epsilon$ we have $\|\frac{1}{N}\sum_{i=mp+1}^{N} F_{i,v}^2 - \frac{q}{N}\|_1 < \epsilon$ hence if $v \ge v(m,\epsilon)$ and $N \ge N_\epsilon$ then

$$(3) \qquad \Big\|1 - \frac{1}{N}\sum_{i=1}^{N} F_{i,v}^2\Big\|_1 = \Big\|1 - \frac{1}{Nv}\sum_{i=1}^{N}\big(\sum_{j=1}^{v} U_{i,j}f\big)^2\Big\|_1 < 2\epsilon.$$

This proves that for $\epsilon > 0$ there are positive integers $v(m,\epsilon/2)$ and N_ϵ such that for $M \ge v(m,\epsilon/2)$ and $n \ge N_\epsilon$, for $X_{n,i} = (1/\sqrt{n})F_{i,M}$

$$\|\sum_{i=1}^{n} X_{n,i}^2 - 1\|_1 = \Big\|\sum_{i=1}^{n}\big(\frac{1}{\sqrt{nM}}\sum_{j=1}^{M} U_{i,j}f\big)^2 - 1\Big\|_1 < \epsilon.$$

In the general case we can suppose that T_{e_d} is ergodic (we can permute the coordinates). Instead of $T_{i,j}$ we will consider transformations $T_{\underline{i},j}$ where $\underline{i} \in \mathbb{Z}^{d-1}$ and in (3), instead of segments $\{km+1,\dots,km+m\}$ we take boxes of $(k_1 m + i_1, \dots, k_{d-1}m + i_{d-1})$, $i_1,\dots,i_{d-1} \in \{1,\dots,m\}$.

This finishes the proof of the Theorem.

□

Remark 2. For any positive integer d there exists a random field of martingale differences $(f \circ T_{\underline{i}})$ for a commuting filtration of $\mathcal{F}_{\underline{i}}$ where $T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, is a non Bernoulli $\mathbb{Z}^d$ action and all T_{e_i}, $1 \le i \le d$, are ergodic.

To show this we take a Bernoulli $\mathbb{Z}^d$ action $T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$ on $(\Omega, \mathcal{A}, \mu)$ generated by iid random variables $(e \circ T_{\underline{i}})$ as defined e.g. in [WaW] or [VWa].
Then we take another $\mathbb{Z}^d$ action of irrational rotations on the unit circle (identified with the interval $[0,1)$) generated by $\tau_{e_i} = \tau_{\theta_i}$, $\tau_{\theta_i}x = x + \theta_i \mod 1$; θ_i, $1 \le i \le d$, are linearly independent irrational numbers. The unit circle is equipped with the Borel σ-algebra $\mathcal{B}$ and the (probability) Lebesgue measure λ.
On the product $\Omega \times [0,1)$ with the product σ-algebra and the product measure we define the product $\mathbb{Z}^d$ action $(T_{\underline{i}} \times \tau_{\underline{i}})(x,y) = (T_{\underline{i}}x, \tau_{\underline{i}}y)$. Because the product of ergodic transformations is ergodic, for every e_i, $1 \le i \le d$, $T_{e_i} \times \tau_{e_i}$ is ergodic. The product $\mathbb{Z}^d$ action is not Bernoulli (it has irrational rotations for factors).

On $\Omega \times [0,1)$ we define a filtration $\mathcal{F}_{(i_1,\dots,i_d)} = \sigma\{U_{(i',\dots,i'_d)} e \circ \pi_1, i' - 1 \le i_1, \dots, i'_d \le i_d, \pi_2^{-1}\mathcal{B}\}$ where π_1, π_2 are the coordinate projection of $\Omega \times [0,1)$.
The filtration defined above is commuting and we can find a random field of martingale differences satisfying the assumptions of the Theorem.

Remark 3. In the one dimensional central limit theorem, non ergodicity implies a convergence towards a mixture of normal laws. This comes from the fact that using a decomposition of the measure μ into ergodic components, we get the "ergodic

case" for each of the components (cf. [V]); the variance is given by the limit of $(1/n)\sum_{i=1}^n U^i f^2$ which by the Birkhoff Ergodic Theorem exists a.s. and in L^1 and is T-invariant. In the case of a $\mathbb{Z}^2$ action (taking $d = 2$ for simplicity), the limit for $T_{0,1}$ need not be $T_{1,0}$-invariant. This is exactly the case described in the Example and eventually we got there a convergence towards a law which is not normal.

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